First passage flows in ecological networks: measurement by input–output flow analysis

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Abstract

This paper explains the rationale behind measuring first passage flows of conservative substances in ecological networks by a simple modification of input–output flow analysis (Higashi et al., 1993a, Ecol. Model., 66: 1–42). Pilette and Kincaid (1992, Ecol. Model., 64: 1–10) posed this “first flow-through” problem and described a different method to solve it. This method, however, does not account for their intended “all non-duplicated routes and transfers” from one compartment to another, and consequently underestimates indirect transfers. In addition, it incorrectly treats storages as equivalent to flows. We correct their version, according to the discrete time input–output flow-storage analysis of Higashi et al. (1993b, Ecol. Model., 66: 43–64), and derive a correct evaluation of direct transfers. This, however, turns out to be equivalent to simple flow ratios obtained from standard, continuous time input–output flow analysis.

Keywords: Networks; Input–output analysis

1. Introduction

Pilette and Kincaid (1992, hereinafter P&K) recently described a “comprehensive data management strategy rather than a matrix approach” – an algorithm – for measuring first passage intercompartmental flows in ecological networks. The method as formulated is incorrect, but a solution to the problem using input–output flow analysis has been obtained by Higashi et al. (1993a). The purpose of this paper is to identify the problems with the P&K approach, and explain the rationale behind the input–output formulation more fully than in the Higashi et al. (1993a) paper.

2. Basic definitions

Let $F = [f_{ij}]$, $f_{jj} = 0$, be an $n \times n$ matrix of time continuous direct flows of substance (carbon in P&K’s example, table 1) from compartments $j$ (columns) to $i$ (rows), $i,j = 1, \ldots, n$, in a network. Let $T_j = \sum_{i=1}^{n} f_{ij}$ be the total flow out of $j$ to
all other compartments $i = 0, 1, \ldots, n$, where 0 denotes the environment. $T_j$ is the throughflow at $j$. If dynamical equations are to be implemented (e.g., Matis and Patten 1981),

$$\frac{dx}{dt} = F1 + z = -F^T1 - y,$$

then the zero diagonal elements of $F$ are replaced by negative throughflows, $f_{jj} = -T_j$. In Eq. 1, $x$, $z$ and $y$ are $n \times 1$ vectors of stocks, inputs and outputs, respectively, for each compartment, and $1$ is an $n \times 1$ vector of ones. $F^T$ is the transpose of $F$. At steady state, $dx/dt = 0$ and $F$ becomes the vehicle for static flow analysis, the case considered by P&K.

In continuous time, mass ($M$)-length ($L$)-time ($T$) dimensions for storages ($x_j$) are $[ML^{-k}]$ and for flows ($f_{ij}$) $[ML^{-k}T^{-1}]$, where $k - 2$ denotes areas (e.g., ha or m$^2$) and $k = 3$ volumes (e.g., m$^3$). If time intervals are discrete, $\Delta t$, the difference equation counterpart of Eq. 1 is

$$x(t + \Delta t) = x(t) + \Delta x(t),$$

where

$$\Delta x = F\Delta t \cdot 1 + z\Delta t.$$  

In this case storages ($x(t)$ and $x(t + \Delta t)$), storage increments ($\Delta x$) and flows ($F\Delta t$ and $z\Delta t$) all have the same dimensions, $[ML^{-k}]$, but the time continuous matrix $F$, with dimensions $[ML^{-k}T^{-1}]$, is still the basis for analysis.

3. The P&K methodology

In the method described by P&K, the difference in storage and flow phenomenology, and hence (in the continuous case) dimensionality, was not mentioned and did not appear to be taken into account. If the diagonal elements in the $5 \times 5$ portion of their table 1 were zeroed, this portion of the table would then represent an $F$ matrix.

These nonzero diagonal elements were stated to be “[s]tocks (transfers in place . . .)” and accordingly have two possible interpretations:

1. If they do represent “transfers in place”, occurring over the loops $j \rightarrow j$ depicted in P&K’s fig. 1, then the proper value in each case should be zero ($f_{jj} = 0, j = 1, \ldots, 5$) since at steady state there is no net transfer from a compartment to itself. Whatever material leaves during a time interval, continuous ($dt$) or discrete ($\Delta t$), must be counterbalanced by a like amount of material which enters. It follows that the diagonal nonzero probabilities (dimensionless fractional transfers, P&K’s table 2), derived from the dimensional diagonal entries in table 1 and used to depreciate stocks sequentially (e.g., top of p. 6: 0.75, 0.75$^2$, etc.), in effect serve as nondimensional initial conditions. The stocks themselves, $x$, must remain constant at steady state, and technically should make no contribution to a flow-only analysis. That P&K used them as initial conditions for a discrete time analysis is shown below. They also included them in their calculation of throughflows.

2. If the diagonal entries in table 1 are “stocks”, or storages, then a discrete time format (with $\Delta t = 1$ day in the authors’ example) is implicitly imposed on the analysis. This is because the stocks in P&K’s table 1, $f_{jj} \equiv x_j$, cannot in the continuous case be either added into or divided by the throughflows (both of which P&K did) to obtain a table 2 whose entries are stated to be dimensionless. That is, the diagonal elements in the $5 \times 5$ matrix in table 2 derived from those of table 1, if time continuous, must have time dimensions since, in the continuous case, $x_j/T_j$ implies $[ML^{-k}]/[ML^{-k}T^{-1}] = [T]$.) For the diagonal elements in table 2 to be dimensionless like the nondiagonal elements, the analysis must be time discrete; only then can $f_{jj} = x_j$ be allowed to contribute to throughflow, and only then does $x_j/T_j\Delta t$ yield nondimensional coefficients since $[ML^{-k}]/[ML^{-k}] = 1$, dimensionless.

Standard input–output flow analysis, which employs continuous time, nondimensionalizes intercompartmental flows as fractions of throughflows, $g_{ij} = f_{ij} / T_j$, which are dimensionless, $[ML^{-k}T^{-1}]/[ML^{-k}T^{-1}] = [1]$, and have values between 0 and 1. Powers of these coefficients, $g_{ij}^{(m)}$, are used to allocate compartmental inputs, $z_{j}$, to paths of different lengths, $m$, corresponding to the powers:

$$g_{ij}^{(m)} \cdot z_j.$$.  

(3)
In this expression the powers are parenthesized to denote that they are obtained by matrix multiplication. The input flows are constant, and it is the nondimensional coefficients themselves that decrease with each higher power computed: $g_{i,j}$, $g_{i,j}^{(2)}$, $g_{i,j}^{(3)}$, \ldots .

For P&K's derivation of direct transfers to make sense, the diagonal entries of their table 1 must be interpreted in a discrete time format, not as stocks per se, but as portions of the stocks in each compartment that remain there during the next time interval, $\Delta t$:

$$x_j - \sum_i x_i f_{i,j} \Delta t = f_{i,j} \Delta t.$$  \hspace{1cm} (4a)

Thus, the column sums (their "compartment totals") equal the stocks (storages) themselves:

$$f_{i,j} \Delta t + \sum_i x_i f_{i,j} \Delta t = (x_j - \sum_i x_i f_{i,j} \Delta t) + \sum_i x_i f_{i,j} \Delta t = x_j.$$  \hspace{1cm} (4b)

This correct interpretation of P&K's table 1, which is in fact equivalent to the interpretation of corresponding elements in input--output flow-storage analysis (Higashi et al., 1993b), derives a correct evaluation of direct transfers by the method P&K described, i.e., by the following expression:

$$p_{ij} \sum_{k=0}^\infty p_{i,j}^k = p_{i,j} \frac{1}{1 - p_{i,j}},$$  \hspace{1cm} (5a)

where $p_{i,j} = f_{i,j} \Delta t / x_j$. This, however, turns out to be reduced to the corresponding fraction of throughflow, $g_{i,j} = f_{i,j} / T_j$, employed in standard input--output flow analysis:

$$p_{ij} \frac{1}{1 - p_{i,j}} = f_{i,j} \Delta t \frac{1}{x_j} \frac{1}{1 - f_{i,j} \Delta t} = f_{i,j} \Delta t \frac{1}{x_j - f_{i,j} \Delta t}$$

$$= \frac{f_{i,j} \Delta t}{\sum_i x_i f_{i,j} \Delta t} = \frac{f_{i,j}}{T_j} = g_{i,j}.$$  \hspace{1cm} (5b)

The indirect transfer along a particular path is, as P&K state, evaluated as the product of direct transfers associated with the links constituting the path, which is thus reduced to multiplying the corresponding fractions of throughflows, $g_{i,j}$. This is what input--output flow analysis already exactly does.

4. Classification of paths

Fig. 1 shows all the groups of paths that can lead from any originating node or compartment, $j$, in a network to any terminating node, $i$. The alpha paths (sets $A$) are simple or acyclic (no repeated nodes) and the omega paths (sets $\Omega$) cyclic. Cyclic paths are simple the first time around. Note that cycles of length 1, e.g., $j \rightarrow j$, are precluded in continuous time flow analysis because self-transfer probabilities are zero (see below: $g_{i,i} = 0$); they do, however, appear in discrete time flow-storage analyses (Higashi et al., 1993b). P&K had to admit such paths to accommodate their nonzero diagonal elements.

First passage paths (set $\Phi$) leading from $j$ to $i$ comprise all the paths in sets $A_{ij}$ and $\Omega_{ij}$, which do not contain cycles around $i$ or $j$, plus those formed by concatenating these sets with the cycles at $j$, $\Omega_{ij}$, which do not touch $i$. That is, a particular first passage path, $\phi_{ij}$, may be one of four categories: $\alpha_{ij}$, $\omega_{ij}$, $\alpha_{ij} \cdot \omega_{ij}$, or $\omega_{ij} \cdot \omega_{ij}$, where "\cdot" means concatenation. Reading subscripts from right to left in these expressions indicates the cardinal nodes involved along each path. Sets of concatenated individual paths will be indicated by shorthand notation, e.g., $A_{ij} \cdot \Omega_{ij}$ and $\Omega_{ij} \cdot \Omega_{ij}$. With this, the entire set of first passage paths from $j$ to $i$ can be represented as the union of the four possible categories: $\Phi_{ij} = A_{ij} \cup \Omega_{ij} \cup A_{ij} \cdot \Omega_{ij} \cup \Omega_{ij} \cdot \Omega_{ij}$. These first passage paths from $j$ to $i$, which do not contain any cycles touching $i$ (but may contain cycles not touching $i$), account for non-cycled network flows, i.e., "all the non-
duplicated routes and transfers” (P&K, p. 2) from \( j \) to \( i \). This is because the event that a mass of substance starting from \( j \) will ever reach \( i \) consists of the nonoverlapping events that a mass starting from \( j \) will arrive at \( i \) for the first time in \( k \) steps, where \( k = 1, 2, \ldots \), and in each of these events the mass follows one of the paths of length \( k \) that do not contain \( i \) (but may contain other nodes) along the way. That is, first passage paths from \( j \) to \( i \) do not contain any cycle touching \( i \), but may contain other cycles not touching \( i \). Any two cases in which a mass starting at \( j \) follows two distinct first passage paths represents two distinct events, because the two cases are separated as the mass comes to the point at which the two paths split. For example, the cases in which the mass follows the two paths \( 1 \to 2 \to 3 \) and \( 1 \to 2 \to 4 \to 2 \to 3 \) become distinct after the mass completes the second transfer. Thereafter, some of the material takes the direction \( 2 \to 3 \) and the remainder the direction \( 2 \to 4 \to \ldots \). P&K restrict to simple paths or cycles their “acceptable paths” available for transfer, excluding many non-duplicated routes that exist from one node to another. Thus, they underestimate indirect transfers, while they would get a correct evaluation for direct transfers (in their table 4, p. 8) if the diagonal entries of their table 1 were interpreted as non-transferred portions of stocks, as we discussed above.

Subsequent passage paths form a cycle nexus at terminal nodes, \( i \). This consists of two path sets: terminal node cycles (set \( \Omega_{ii} \)) which do not pass through the originating compartment, \( j \), of an \((i,j)\) pair; and feedback cycles (set \( \Phi_{ii} \)), which do, and contain feedback paths, \( \alpha_{ji} \) or \( \omega_{ji} \), and first passage paths. Thus, in terms of path sets, \( \Phi_{ii} = [(A_{ij} \cup \Omega_{ij}) \cdot (A_{ji} \cup \Omega_{ji})]^* \), where \( * \) represents any number of repetition of the bracketed sets and the cycle nexus at \( i \) is \( \Omega_{ii} \cup \Phi_{ii} \). These paths will be shown later to account fully for the cycled flows in networks.

The set, \( \Psi_{ij} \), of total paths from \( j \) to \( i \) consists of first passage paths and their concatenation with subsequent passage paths: \( \Psi_{ij} = \Phi_{ij} \cup (\Omega_{ij} \cup \Phi_{ji}) \cdot \Phi_{ij} \). Patten (1982, 1985) has shown that combinatorially large numbers of such paths exist in networks with even small numbers of compartments. For example, a 24-compartment marsh food web model for Okefenokee Swamp, USA, with 21\% connectivity and 45\% cycling, has over 44-million simple paths (those in the alpha sets \( A_{ij}, \forall i,j \)) alone. Numbers like these will overwhelm methods (like P&K’s) based on particular paths, but to understand the underlying principles involved it is necessary to explore small number cases.

Input–output flow analysis accounts for the flows over the entire path set, \( \Psi_{ij} (\forall i,j) \), of all types and lengths, and a simple modification of the standard methodology enables the first passage flows, those associated with paths \( \Phi_{ij} \), to be computed.

5. Input–output flow analysis

The first flow-through analysis problem posed by P&K (1992) has been solved by matrix methods from standard input–output flow analysis (Higashi et al., 1993a).

The sixth row of P&K’s table 1, labeled “Out”, denotes dissipative losses and, transposed, corresponds to their model’s \( 5 \times 1 \) output vector: \( y_{\Delta t} = [f_{0j}\Delta t] = [22114]^T \), in g \( C \), \( j = 1, \ldots, 5 \). Note that the real units are “daily flows in grams of carbon” (table 1 legend, p. 3), i.e., g \( C/\text{day} \). However, these latter units are only appropriate to a time continuous analysis and, as explained above, P&K’s treatment of storages (rather than, more correctly, their derivatives or discrete time rates of change) as flows has, as required for dimensional consistency, implicitly imposed a discrete time format on their method. The seventh (bottom) row, labeled “Compartment total”, should therefore be considered as stocks, as explained above. Removing the diagonal elements (nontransferred portions of stocks) from these yields throughflows, \( T_{\Delta t} = [T_{j}\Delta t] = [108436] \), in g \( C \). The sum of these is the total system throughflow, \( TST_{\Delta t} = 31 \) g \( C \). Input is to the first compartment only, \( z_{\Delta t} = 10 \) g \( C \); the input vector is thus \( z = [f_{0j}] = [1000000]^T \). Although central in flow analysis as the source of internal flows, input is not used in P&K’s analysis except to balance outflows against in compartment 1.
In standard, continuous time, input-output flow analysis, a dimensionless \( n \times n \) flow intensity matrix \( G = [g_{ij}] \), \( g_{jj} = 0 \), where \( n \) is the number of compartments, is formed from \( F \) by normalizing the dimensional flows with respect to throughflows: \( g_{ij} = \frac{f_{ij}}{T_j} \). The matrix power series

\[
I + G + G^2 + G^3 + G^m + \ldots = (I - G)^{-1} = N
\]

accounts for all nondimensional flows derived from unit initial flows, \( I \) (the identity matrix), and subsequently propagated through the network over all cyclic and noncyclic paths of all lengths, \( m = 1, 2, \ldots \). The acyclic paths can be no longer than \( m = n - 1 \) because the next increment in path length must then close a cycle since all compartments would already have previously been visited. The dimensionless convergent matrix in Eq. 6, \( N = [n_{ij}] \), takes the dimensional steady state input vector, \( z = I_z \), into the steady state throughflow vector, \( T \):

\[
T = Nz.
\] (7a)

This implicitly accounts for all the dimensional flows over all the paths, \( \Psi_{ij} \), of all types (Fig. 1) and lengths \( (m = 1, 2, \ldots, \infty \) in the limit) directed from each \( j \) to each \( i \) in the network. That is, \( n_{ij}z_j \) gives the total steady state flow at compartment \( i \) derived from input to \( j \); this represents a fraction, \( T_{ij} \), of the total throughflow at \( i \): \( T_{ij} = n_{ij}z_j \); remaining fractions of \( T_i \) are derived from inputs to other compartments. Hence,

\[
T_i = \sum_{i=1}^{n} T_{ij} = \sum_{i=1}^{n} n_{ij}z_j.
\] (7b)

Note the dimensional consistency here since each of these expressions has the flux dimensions of time continuous flows, \([ML^{-k}T^{-1}]\).

The terms of the series expansion of \( N \) in expression 6 partition the total dimensional flow over all paths of all types, and of lengths \( m = 0, 1, 2, \ldots \), from each specified \( j \) to each specified \( i \). The \( g_{ij}^{(m)} \) coefficients in these terms give the nondimensional flows. In P&K’s model, for example, nondimensional flows associated with path \( 1 \rightarrow 2 \rightarrow 3 \) would be accounted for in element \( g_{51}^{(2)} \) of the \( G^2 \) matrix, those due to path \( 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \) would appear in element \( g_{51}^{(4)} \) of the \( G^4 \) matrix, and those generated by paths \( 5 \rightarrow 2 \rightarrow 3 \rightarrow 5 \) and \( 5 \rightarrow 2 \rightarrow 4 \rightarrow 5 \) would both be summed together in element \( g_{55}^{(3)} \) of the \( G^3 \) matrix. Thus, there is consistency between the quantitative analysis of flows afforded by input-output theory and the path structure of models.

### 6. Cycling and noncycling indexes

The Finn (1976a,b) cycling index quantifies the degree of cycling in the flow structure of a network. It is based on the diagonal elements of the \( N \) matrix, whose elements in general, \( n_{ij} \), in addition to taking inputs into throughflows as defined by Eq. 7a, have an alternative interpretation as the expected number of times a unit of substance derived from source \( j \) will appear in terminal \( i \) during its residence time in the system (Kemeny and Snell, 1960; Matis and Patten, 1981). For an alternative approach to obtaining the same cycling index, see Szyrmer and Ulanowicz (1987).

The diagonal elements, \( n_{ii} \), have the same interpretation. Their values are always \( > 1 \). When \( n_{ii} = 1 \), enabling \( 1/n_{ii} \) (the noncycling coefficient; see below) to equal 1 also, this denotes no cycling, i.e., the restriction of flow to first passage sequences only. When \( n_{ii} > 1 \), cycling is indicated and quantified by \( n_{ii} - 1 \), the expected number of revisits after the initial first passage transfer.

Cycled throughflow in the cycle nexus at each \( i \) is computed by multiplying the total throughflow, \( T_i \), by the ratio \( (n_{ii} - 1)/n_{ii} \). Summation of these values over all the terminal nodes yields the total system cycled throughflow, \( T_{ST} \):

\[
T_{ST} = \sum_{i=1}^{n} \left[ \frac{n_{ii} - 1}{n_{ii}} \right] T_i.
\] (9)
Similarly, the coefficient for noncycled or first passage throughflow at each \( i \) is formed as \( 1 - [(n_{ji} - 1)/n_{ii}] = 1/n_{ii} \), as observed above. Using this, the total system first passage or uncycled throughflow, \( \text{TST}_\Phi \), can be calculated:

\[
\text{TST}_\Phi = \sum_{i=1}^{n} \left[ \frac{1}{n_{ii}} \right] T_i.
\]  

(10)

Total system throughflow, \( \text{TST} \), is then the simple sum of Eqs. 9 and 10:

\[
\text{TST} = \text{TST}_\Phi + \text{TST}_\Omega.
\]  

(11)

The Finn cycling index, \( \text{CI} \), is the ratio of cycled to total system throughflow:

\[
\text{CI} = \frac{\text{TST}_\Omega}{\text{TST}}.
\]  

(12a)

Finally, a noncycling or first passage index, can similarly be defined:

\[
\text{CI} = \frac{\text{TST}_\Phi}{\text{TST}}.
\]  

(12b)

With these developments, it is possible to solve the problem of determining first passage flows in networks by a simple modification of standard input–output methodology.

7. First passage flows by input–output analysis

Eqs. 7 give the essential relations which take environmental inputs to compartments into throughflow contributions. The coefficients that do this from each \( j \) to each \( i \) are \( n_{ij} \), and these account for all the flows in the system over the entire path set \( \Psi_{ij} \) (\( \forall i,j \)).

The throughflows are then partitioned in Eqs. 9 and 10 into cycled and noncycled portions. The first passage and cycling coefficients which achieve this are, respectively, \( 1/n_{ii} \) and \( (n_{ii} - 1)/n_{ii} \). The fact that these sum to 1 reflects their status as partition coefficients. When \( 1/n_{ii} = 1 \), as noted above, the cycling coefficient is zero denoting no cycling. The partition property enables both first passage and cycling coefficients to be formulated which will map dimensional inputs into noncycled and cycled throughflows, respectively. These coefficients are \( n_{ij}/n_{ii} \) for first passage, and \( n_{ij}(n_{ii} - 1)/n_{ii} \) for cycling.

Thus, the input–output flow analysis solution to P&K's first flow-through problem is, from Eq. 7:

\[
T_\Phi = N_\Phi z = [n_{ij}/n_{ii}] z
\]  

(Higashi et al., 1993a), where \( N_\Phi = [n_{ij}/n_{ii}] \) is the \( n \times n \) matrix of \( n_{ij}/n_{ii} \) coefficients, and in scalar form:

\[
T_{i,\Phi} = \sum_{j=1}^{n} T_{ij,\Phi} = \sum_{j=1}^{n} (n_{ij}/n_{ii}) z_j, \quad i = 1, \ldots, n.
\]  

(13b)

The corresponding formulations for cycled flows are:

\[
T_\Omega = N_\Omega z = [n_{ij}(n_{ii} - 1)/n_{ii}] z
\]  

and

\[
T_{i,\Omega} = \sum_{j=1}^{n} T_{ij,\Omega} = \sum_{j=1}^{n} [n_{ij}(n_{ii} - 1)/n_{ii}] z_j,
\]

\( i = 1, \ldots, n \).  

(14b)

Expressions 13 represent the throughflows at each compartment in a network associated with the first passage path set, \( \Phi_{ij} \) (\( \forall i,j \)), and Eqs. 14 denote the throughflows generated by subsequent passages around the cycle nexuses, \( \{\Omega_{ij} \cup \Phi_{ij}\} \) (\( \forall i \)). Here, however, for \( j = i \), \( n_{ij} \) should be replaced by \( n_{ii} - 1 \).

8. Application to P&K's model

Table 1 shows the \( F \) matrix, in continuous time input–output analysis, for the P&K example model, as modified and reformatted from their table 1; also shown is the \( G \) matrix. The first column and bottom row of \( F \) display, respectively, input (\( z \)) and output (\( y \)) vectors. Row and column sums give elements of the throughflow vector (\( T \)); these sum to the total system throughflow, \( \text{TST} = 31 \text{ g C} \).

Table 2 shows the \( N \) and \( [n_{ij}/n_{ii}] \) matrices. The diagonal entries in \( N \) contain the cycling information. The only compartment outside the
cycle nexus (P&K fig. 1, p. 3) is 1 since \( \frac{n_{11} - 1}{n_{11}} \) = 1.000. The others experience cycled flow, so the \( \frac{n_{ii} - 1}{n_{ii}} \) coefficients for compartments 1–5 are, respectively, 0.000, 0.167, 0.100, 0.091, and 0.167. The mean of these values is 0.106, indicating 10.6% nondimensional cycling. The products of these coefficients and throughflows of their respective compartments, \( T_i \frac{n_{ii} - 1}{n_{ii}} \), are: 0.000, 1.322, 0.418, 0.272, and 0.999. These sum to a total system cycled throughflow of \( T_{ST} = 3.021 \) g C. The system cycling index for dimensional flows is thus \( CI = \frac{T_{ST}}{T_{ST}} = 0.098 \), or 9.8%. The first passage index is the additive reciprocal, \( CI = 1 - 0.098 = 0.902 \), indicating that 90.2% of carbon flow in the P&K system is uncycled.

This result can be used to verify the first passage formulation of the previous section, Eqs. 13. Elements of the \( \left[ \frac{n_{ij}}{n_{ii}} \right] \) matrix in Table 2, with diagonal entries \( \frac{n_{ii} - 1}{n_{ii}} \) for \( j = i \), represent nondimensional flows over all acyclic and cyclic paths in the first passage path set, \( \Phi_{ij} (\forall i,j) \). Multiplying this matrix by the input vector (Eq. 13a) yields the vector of first passage throughflows: \( T_\Phi = [10.000 \ 6.666 \ 3.581 \ 2.727 \ 5.001]^T \). The sum of these entries gives the first passage total system throughflow, \( T_{ST} = 27.975 \) g C/day. The ratio of this to the total system throughflow is \( 29.975/31 = 0.902 \), which corresponds to above. This verifies Eq. 13 for the measurement of first passage flows, and demonstrates the correctness of the input–output formulation for solving P&K’s first flow-through problem.

### Table 1

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<th>3</th>
<th>4</th>
<th>5</th>
<th>( T )</th>
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</tr>
<tr>
<td>4</td>
<td>0.273</td>
<td>0.409</td>
<td>0.534</td>
<td>1</td>
<td>0.136</td>
<td>0</td>
</tr>
<tr>
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<td>0.500</td>
<td>0.500</td>
<td>0.584</td>
<td>0.667</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

\( F \) matrix plus input \( (z) \), output \( (y) \), and throughflow \( (T) \) vectors, and \( G \) matrix for the carbon flow model of P&K. Flows, in g C/day in \( F \) and dimensionless in \( G \), are from columns to rows.
change (finite difference case) are. P & K ignored or were unaware of this, and consequently treated storages and flows as interchangeable. The result is that in the continuous case their dimensions do not even balance. For P & K’s derivation of direct transfers to make sense, the diagonal entries of their table 1 must be interpreted, in a discrete time format, not as stocks per se, but as portions of the stocks in each compartment that remain there during the next transfer interval. Thus, the column sums equal the stocks themselves. This corrected version, which would then be equivalent to the input–output flow-storage analysis of Higashi et al. (1993b), derives a correct evaluation of direct transfers by the method that P & K described, which, however, turns out to be equivalent to simple flow ratios (the G matrix) employed in standard input–output flow analysis. This can be verified by comparing the (dimensionless) direct transfers given in P & K’s table 4 with the G matrix of Table 1 here. They give the same values except for the diagonal elements, which must by definition be zero, though P & K inconsistently equate these elements with the ratios of their “stocks” over the “compartment totals”, the diagonal elements of their table 2.

References